UNIVERSITY OF CALCUTTA · Previous Year Question Paper

2025

MATHEMATICS — HONOURS

Paper: DSCC-8

(Group Theory - II and Ring Theory - I)

Full Marks: 75

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The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
(All symbols used have their usual meanings.)

Group - A

[Group Theory - II]
(Marks : 40)

Answer *any eight* questions.

1. Let H be a subgroup of a group G such that $x^2 \in H$, for all $x \in G$. Show that H is a normal subgroup of G and G/H is commutative. **3+2**

2. If ϕ be a homomorphism of a group G into a group G', show that

(i) $\phi(a^{-1}) = \{\phi(a)\}^{-1}$, for any $a \in G$

(ii) $\phi(a^n) = \{\phi(a)\}^n$, for any integer n and any $a \in G$. **2+3**

3. State and prove Cayley's theorem for a group. **5**

4. Let G be a group and H, K be subgroups of G such that G is an internal direct product of H and K. Show that

(i) $G \simeq H \times K$ and

(ii) $G/H \simeq K$. **3+2**

5. Let G be a multiplicative group of order 10, and let $f : G \to G$ be a mapping defined by $f(x) = x^9$, $\forall x \in G$. Show that if G is commutative, then f is an automorphism. Is the converse true? Justify your answer. **3+2**

6. (a) If G be a non-commutative group with centre $Z(G)$, then prove that $G/Z(G)$ is non-cyclic.

(b) Prove that a non-abelian group of order 10 must have a trivial centre. **3+2**

7. Let G and G' be two groups and $\varphi : G \to G'$ be an epimorphism. Prove that $G/\operatorname{Ker}\varphi$ is isomorphic to G'. **5**

8. Let H, K be two finite cyclic groups of order m, n respectively. Prove that $H \times K$ is cyclic group if and only if $\gcd(m, n) = 1$. **5**

9. (a) Let G be a cyclic group of order n. Prove that $|\operatorname{Aut}(G)| = \phi(n)$, where ϕ is the Euler's phi function.

(b) Let G be a group such that $\operatorname{Aut}(G) = \{I_G\}$. Prove that G is a commutative group and $g^2 = e$, $\forall g \in G$. **3+2**

10. Let G be a finite abelian group of order n and m be a positive divisor of n. Prove that G has a subgroup of order m. **5**

11. Let H and K be two subgroups of a group G. If H is normal in G, prove that HK is a subgroup of G. Furthermore, if K is normal in G, then prove that HK is also normal in G. **5**

Group - B

[Ring Theory - I]
(Marks : 35)

Answer *any seven* questions.

12. Show that a non-trivial finite ring having no divisor of zero is a ring with unity. **5**

13. Define characteristic of a ring. Prove that the characteristic of an integral domain is either zero or prime number. **5**

14. (a) Let a, b be two elements in a field F with $b \neq 0$. If $(ab)^2 = ab^2 + bab - b^2$, prove that $a = 1$.

(b) Let R be an integral domain and $a \neq 0$ be an element of R. If $na = 0$ for some integer n, show that R is of finite characteristic. **3+2**

15. Show that a finite integral domain is a field. **5**

16. (a) Prove that $S = \{(a, 3b) : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{Z} \times \mathbb{Z}$.

(b) Give an example of a finite ring R with unity and a subring S of R having no unity with proper justification. **3+2**

17. Define ideal of a ring. Let R be a ring and $a \in R$. Prove that $S = \{r \in R : r.a = 0\}$ is a left ideal of R. Give example to establish that it may not be a right ideal of R. **1+3+1**

18. (a) For positive integer n, prove that the ring $\mathbb{Z}_n$ is isomorphic to the ring $\mathbb{Z}/n\mathbb{Z}$.

(b) For the ring homomorphism $f : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $f(a_0 + a_1x + \cdots + a_nx^n) = a_0$, find Ker f. **3+2**

19. Let R be a commutative ring with unity. Prove that an ideal I is maximal if and only if R/I is a field. **5**

20. Prove that a commutative ring with unity has no non-trivial ideals if and only if it is a field. **5**

21. (a) Let ρ be a congruence on a ring R. Prove that $I = \{r \in R \mid r\rho o\}$ is an ideal of R.

(b) Prove that $5\mathbb{Z}$ is a maximal ideal of the ring $\mathbb{Z}$. **3+2**

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