UNIVERSITY OF CALCUTTA · Previous Year Question Paper

2025

MATHEMATICS — HONOURS

Paper: DSCC-10

(Ring Theory - II and Linear Algebra - I)

Full Marks: 75

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The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Ring Theory - II]
(Marks : 40)

1. Answer *any five* questions : 3×5

(a) If F is a field, then prove that $F[x]$ is an Euclidean domain.

(b) Prove that g.c.d. of $\bar{4}$ and $\bar{6}$ in $(\mathbb{Z}_8, +, \cdot)$ exists but not unique.

(c) In $\mathbb{Z}[\sqrt{5}]$ prove that 2 is irreducible but not prime.

(d) Use Eisenstein's criterion to show that $f(x) = 2x^4 + 6x^3 - 9x^2 + 15$ is irreducible over $\mathbb{Z}$.

(e) Determine all associates of $8 + 3i$ in $\mathbb{Z}[i]$.

(f) Factorize $x^2 + 3$ as a product of irreducible elements of $\mathbb{Z}_7[x]$.

(g) Let F be a field. Prove that a polynomial of degree 2 or 3 in $F[x]$ is irreducible if and only if it has no root in F.

(h) Prove that every field is a regular ring.

Answer *any five* questions :

2. (a) Define an Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. **1+4**

(b) In a principal ideal domain R, show that for any two non-zero elements $a, b \in R$, $m = \text{l. c. m.}(a, b)$ if and only if $\langle m \rangle = \langle a \rangle \cap \langle b \rangle$. **5**

(c) Prove that $[2]$ is a prime element in $\mathbb{Z}_{10}$ but $[2]$ is not irreducible in $\mathbb{Z}_{10}$. **2+3**

(d) Define a unique factorization domain. Prove that in a unique factorization domain every irreducible element is prime. **1+4**

(e) Prove that the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Z}[x]$, where p is a prime number. **5**

(f) If R is a commutative ring with unity such that $R[x]$ is P.I.D, then show that R is a field. **5**

(g) Prove that any ring (with more than one element) which is commutative and without non-zero divisors of zero can be embedded in a field. **5**

(h) (i) Show that a ring R with identity is a regular ring if and only if for any left ideal A and right ideal B of R, $BA = A \cap B$.

(ii) Prove or disprove : $\mathbb{Z}$ is a regular ring. **4+1**

Group - B

[Linear Algebra - I]

(Marks : 35)

3. Answer *any five* questions : 3×5

(a) Find a basis for the vector space $\mathbb{R}^3$ that contains the vectors $(1,2,0)$ and $(1,3,1)$.

(b) Find the dimension of the subspace S of the vector space $\mathbb{R}^3$ defined by : $S = \{(x,y,z) \in \mathbb{R}^3 : x+2y=z,\ 2x+3z=y\}$

(c) Find the co-ordinate vector of α in $\mathbb{R}^3$ relative to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha = (0,3,1)$, $\alpha_1 = (1,1,0)$, $\alpha_2 = (1,0,1)$, $\alpha_3 = (0,1,1)$.

(d) Find a basis for the row space of the matrix $A = \begin{pmatrix} 0 & 3 & 7 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

(e) Show that two eigen vectors of a square matrix A over a field F corresponding to two distinct eigenvalues of A are linearly independent.

(f) If A and P be both $n \times n$ matrices and P be non-singular, then prove that A and $P^{-1}AP$ have same eigenvalues.

(g) Let $T : V \to W$ be a linear mapping where V are W are finite dimensional vector spaces of same dimension over a field F. If T is surjective (onto), then prove that T is one-to-one.

(h) In a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$, $[T] = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{2}{3} & 4 \end{bmatrix}$ associated with the basis $\{(1,0),(1,1)\}$. Find $T(x,y)$.

Answer *any four* questions :

4. (a) Let V be a finite dimensional vector space over a field F_1 and W be a subspace of V. Show that $\dim(V/W) = \dim V - \dim W$. **5**

(b) When is a set of vectors $\{x_1, x_2, \ldots, x_n\}$ said to be linearly independent? Show that the set of vectors $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent in $\mathbb{R}^3$. **1+4**

(c) Find a basis and find dimension of the vector subspace W of $\mathbb{R}^3$ where $W = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0,\ 2x+y+3z=0\}$. **3+2**

(d) Let $AX = 0$ be a homogeneous system of equations in n unknowns and $X(A)$ be the solution space of the system of equations. Then prove that rank of A + rank of $X(A) = n$. **5**

(e) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases

$$\{(0,1,1),(1,0,1),(1,1,0)\} \text{ of } \mathbb{R}^3 \text{ and } \{(1,0),(1,1)\} \text{ of } \mathbb{R}^2$$

is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of T relative to the ordered bases $\{(1,1,0),(1,0,1),(0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1),(0,1)\}$ of $\mathbb{R}^2$. **5**

(f) Prove that eigen values of a real symmetric matrix are all real. **5**

(g) State Cayley-Hamilton Theorem and verify it for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$. Hence find out A^{-1}, if exists. **1+3+1**