UNIVERSITY OF CALCUTTA · Previous Year Question Paper

2025

MATHEMATICS — HONOURS

Paper: DSCC-11

(Riemann Integration and Series of Functions)

Full Marks: 75

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The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Throughout the question $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of real numbers, rational numbers and set of natural numbers respectively.

Group - A
(Marks : 50)

1. Answer *any five* questions : 3×5

(a) Let $f : [0,4] \to$ be defined by $f(x) = x^4 - 4x^3 + 10$ and $P = \{1,2,3,4\}$. Find $U(P,f)$ and $L(P,f)$.

(b) Prove that, $\mathbb{Q}$ is a negligible set.

(c) Let $f(x) = [x]; 0 \le x \le 3$. Evaluate $\int_0^3 f$.

(d) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} (\sin 2x)^n$. Check whether f is Riemann integrable on $[0,1]$.

(e) If f is continuous on $[a,b]$, $f(x) \ge 0$ on $[a,b]$ and $\int_a^b f = 0$, then prove that $f(x) = 0$ for all x on $[a,b]$.

(f) Let $f(x) = \begin{cases} c; & 0 \le x \le c \\ 2c; & c < x \le 1 \end{cases}$. If $\int_0^1 f = \frac{7}{16}$, then find the value of c.

(g) Examine the convergence of $\int_1^2 \frac{\log x}{\sqrt{2-x}}\, dx$.

(h) Find the value of $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Answer *any seven* questions :

2. (a) Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be a partition of $[a,b]$. If Q is a refinement of P, then prove that $U(P,f) \ge U(Q,f)$ and $L(P,f) \le L(Q,f)$. **5**

(b) State and prove a necessary and sufficient condition for Riemann integrability of a bounded function f defined on $[a,b]$. **1+4**

(c) (i) Let f, g, h be three bounded functions on $[a,b]$, such that $f(x) \le g(x) \le h(x), \forall x \in [a,b]$. If $f(x)$ and $h(x)$ are Riemann integrable and $\int_a^b f(x)\, dx = \int_a^b h(x)\, dx$, show that g is also Riemann integrable on $[a,b]$.

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(ii) Find the number of points of discontinuities of the function $\phi(x) = \int_0^x \left[\sqrt{t}\right] dt, 0 \le x \le 2026$, ($[r]$ denotes greatest integer $\le r$). **3+2**

(d) If $f, g : [a,b] \to \mathbb{R}$ are Riemann Integrable on $[a,b]$, show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ (Assume that $f+g$ is also Riemann Integrable on $[a,b]$). **5**

(e) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then prove that the function $F(x) = \int_a^x f(t)\,dt$; $a \le x \le b$ is differentiable on $[a,b]$. **5**

(f) Let f be a differentiable function on $[a,b]$ with $|f'(x)| \le C, \forall x \in [a,b]$ for some $C > 0$. Prove that for any partition P of $[a,b]$, $U(P,f) - L(P,f) \le C\|P\|(b-a)$. **5**

(g) Prove that $\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}, k^2 < 1$. **5**

(h) Let f be defined on $[-2,2]$ by $f(x) = \begin{cases} 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that f is integrable on $[-2,2]$. Evaluate $\int_{-2}^{2} f$, using fundamental theorem of integral calculus. **1+4**

(i) Show that the improper integral $\int_0^1 \frac{1}{x} \sin\frac{1}{x}\,dx$ is convergent. **5**

(j) (i) Prove that $\int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx$ is convergent if and only if $n < 1+m$.

(ii) Find the value of $\int_0^\infty x^2 e^{-x^4} dx$. **3+2**

(k) State Dirichlet's test for improper integral. Applying this show that $\int_0^1 x^{-3/2} \sin\left(\frac{1}{x}\right) dx$ is convergent. **2+3**

Group - B
(Marks : 25)

3. Answer *any five* questions :

(a) Prove that a sequence $\{f_n\}$ of real-valued functions defined in $E \subset \mathbb{R}$ converges uniformly in E if and only if for every $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $\forall m, n > k$ and $\forall x \in E$, $|f_n(x) - f_m(x)| < \epsilon$. **5**

(b) Show that the sequence $\left\{\frac{nx}{1+n^2x^2}\right\}_n$ of continuous functions defined on $[0,1]$ has a continuous limit function. Also show that the convergence is not uniform. **2+3**

(c) Let $f_n(x) = \frac{\log(1+n^2x^2)}{n^2}$, $x \in [0,1]$. Show that $\{f_n\}$ is uniformly convergent on $[0,1]$. **5**

(d) Show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$ is uniformly convergent on $\mathbb{R}$. Hence find $\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$. **2+3**

(e) Examine the uniform convergence of $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots$, where $0 \le x \le 1$. **5**

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(f) Assuming $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots; (-1 < x < 1)$, obtain the power series expansion of $\log_e(1+x)$ and find the region of convergence of the power series of $\log_e(1+x)$. **3+2**

(g) (i) Prove or disprove : Radius of convergence of a power series remains invariant under term-by-term differentiation.

(ii) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, where for all $n \in \mathbb{N}$, $a_n =$

$$\begin{cases} \frac{1}{3^n}, & \text{if } n \text{ is odd} \\ \frac{1}{2^n}, & \text{if } n \text{ is even.} \end{cases}$$

3+2

(h) Find the Fourier series for a periodic function $f : \mathbb{R} \to \mathbb{R}$ of period 2π defined by $f(x) = |x|$ for $-\pi \leq x \leq \pi$. **5**

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